A Comparative Study of Two Zitterbewegung Formulations in Quantum Theory?

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In a previous note [2], we argued that the formalism of quantum mechanics follows directly from Einstein´s 1905 equation $p^2 + m^2 = E^2$ (1) with the introduction of a diagonal 2x2 matrix with +E and -E and eigenvalues. (Here we are using a one dimensional model and only techniques from linear algebra.)

An equivalent energy matrix H was shown to be (with c set to 1):

|p m| (2). We showed that a velocity matrix V |1 0 | (3)
|m-p| |0-1 |

immediately followed with no use of commutation relations between p and x as in the Dirac case. We also showed that the evolution operator is exp(-iHt), so dO/dt = [H,O] as in quantum mechanics and that the Zitterbewegung equation of Schrodinger, as listed in [1], follows. The purpose of this second note is to show that there appears to be a second, internal zitterbewegung or at the very least, a different approach to calculating zitterbewegung which uses an explicit time independent matrix for X. This follows from the fact that an X operator matrix only exists for the case of p=0 in H (2), i.e. in the center of mass frame. The velocity matrix V, however, exists in both the p=0 and p>0 frames. In the p=0 frame, its expectation in 0 and in the p>0 frame, its expectation is p. As a result, one can derive Schrodinger zitterbewegung for an x which represents a particle´s motion in space, together with a second zitterbewegung which describes internal motion. For p=0, the two are the same.

The First Type of Zitterbewegung Formalism

To begin, let us consider the velocity vector (3). In [2], it was noted that pv+ sqrt(1+ v2)m=E (4) as a scalar equation. Comparing with (2), we identified a matric V with the scalar v in (4). In Dirac formalism, the v operator is found using [H, x], with p proportional to d/dx. We use no such assumption here. Then, V(t) = exp(iHt) V exp(-iHt) [1] and dV/dt = [H, v]. We also found in [2] that dO/dt=[H,v]. Thus, one should try to write: V=[H,X] (5) because V is given as a matrix by (3) and H by (2). There is no matrix solution in our one dimensional model, however, to this equation (5) (except in the p=0 case as we shall examine later.)

Trying to solve (5) with X = |a b| gives:
|c d|

$m(c-b)=1$ $m(c-b)=-1$ $2pb + m(d-a)=0$ and $-2pc + m(a-d) = 0$.

In order to have consistency, i.e. to have (5) be true, we can say that $[p,x]=1$, in order to ¨pick off¨ the V matrix from (2). Now, p is a parameter, namely the particle momentum, so x should be d/dp. This is equivalent to the formalism in which x is a variable and p is proportional to the x derivative. As a result, the $[p,x]=1$ formalism appears, although it is introduced in a somewhat forced manner. Without it, however, there is no solution to (5).

After this we perform similar calculations as in [1].

We have V given by a matrix (3). We have $V(t)= \exp(iHt)\, V(0)\, \exp(-iHt)$ with H given by (2). Next, we calculate $[H,V] = dV/dt = \begin{vmatrix} 0 & -2m \\ 2m & 0 \end{vmatrix}$.

We find that $dV/dt = 2(pI- VH)$ where I is the identity matrix and p the particle momentum parameter. Our results are identical to the Zitterbewegung results of [1], so we simply list the solution of [1].

$X(t) = X(0) + p H^{-1} + (½)\, i H^{-1} (V- H^{-1}) (\exp(2iHt -1)$ or

$X(t) = X(0) + p H^{-1} + (½)\, i H^{-1} (V-H^{-1}) (\cos(Et)I - i (1/E) \sin(Et)H)$ (6)

Here X(t) represents the particle position, so we see a linear term analogous to d=vt. (Note: p/E=v.)

Second Kind of Zitterbewegung Formalism

It is interesting to consider that the velocity matrix V holds in the case of p=0. In such a case, the energy matrix becomes $M= \begin{vmatrix} 0 & m \\ m & 0 \end{vmatrix}$. This has the eigenvector $(1/\mathrm{sqrt}(2))\ (1,1)$.

Now, the V matrix (3) still holds with $<V>=0$ and its eigenvectors are (1,0) and (0,1). Thus, the positive energy eigenvalue for p=0 is a combination of positive and negative velocity eigenvectors. In addition, a momentum matrix is: $P=\begin{vmatrix} p & m \\ m & -p \end{vmatrix}$ which is $\begin{vmatrix} 0 & m \\ m & 0 \end{vmatrix}$ for p=0.

One can see that $[M,P]=0$ for p=0, so the momentum and energy states commute, and in fact have the same eigenvalues. This reminds one of a photon which has energy equal to momentum. In fact, a trapped photon would represent energy equal to m in a localized position.

Let us now try to derive a general X operator using $[M,X] = V$ (4) with V given by (3). Image that there is a matrix B which satisfies (4), then a general solution is B+ D, where D is the most general matrix which commutes with M. We find:

$X= aI + d\, |0 \quad 1| + |0 \quad b|$ where I is the identity matrix and a and d constants

|1 -2p/m| |b+1/m |

Now, the second matrix is a general matrix which commutes with H. Setting p=0, we just get a matrix proportional to M.   The third matrix is a specific solution of (4). We can rewrite the above result as:

X= aI   +   dM   + |0   0| (7)   where a and d are arbitrary constants.
|1/m 0|

The expectation value for X with the velocity eigenvector (1,0) is ¨a¨ and with the energy eigenvector 1/sqrt(2) (1,1),   ¨a+dm + 1/m

Now [M,X]=V   and   X(t) = exp (iMt) X exp(-iMt) or:

$X(t) = \cos^2(mt)\,X - i\,(1/m)\cos(mt)\sin(mt)\,XM + i\,(1/m)\sin(mt)\cos(mt)\,(XM+V)$
$+ (1/m^2)\sin^2(mt)\,(XM+V)M$ note: MX= XM + V   (8)

Here, in the second zitterbewegung there is no term linear in t.

Now: V(t) = exp(iMt) V(0) exp(-iMt)   with V(0) given by (3).

Then: $V(t)= \cos^2(mt)V - i\,(1/m)\cos(mt)\sin(mt)\,VM + i(1/m)\cos(mt)\sin(mt)\,MV +$
$(1/m^2)\sin^2(mt)\,MVM.$ (7)

Now, let us calculate:

$V(t) = [M, X(t)] = \cos^2(mt)\,V - i(1/m)\cos(mt)\sin(mt)\,VM + i(1/m)\cos(mt)\sin(mt)\,(VM + dV/dt)$
$+ \quad (1/m^2)\sin^2(mt)(VM+dV/dt)M$ with VM + dV/dt = MV.

We obtain consistent results.

Comparison of the Two Zitterbewegung Expressions

The second zitterbewegung expression (8) only holds for p=0. It is only in this case that there is an X matrix such that [M,X] = v. If p>0, it seems that this zitterbewegung should occur in the center of mass system i.e. internally, while the first case should be zitterbewegung of the particle position. (One might feel that these are one and the same, obtained using different formalisms. If that is the case, it seems that some information is lost moving from an X matrix to an x variables which satisfies [p,x]=1.

If one considers the expression for the first zitterbewegung (6) with p=0 one has:
$X(t)=X(0)+ i(½)H^{-1}\,V\,(\cos(2mt)\,I - i\,(1/m)\sin(2Et)H-1)$

$$H^{-1} = (1/m)\begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} \quad \text{so} \quad X(t)-X(0) = (1/m)\begin{vmatrix} i\cos(mt)\sin(mt) & \sin^2(mt) \\ -\sin^2(mt) & -i\cos(mt)\sin(mt) \end{vmatrix} \quad (9)$$

Expression (8) yields X(t) time part =

$$(1/m)\begin{vmatrix} i\cos(mt)\sin(mt) & \sin^2(mt) \\ \cos^2(mt) & -i\cos(mt)\sin(mt) \end{vmatrix} \quad (10)$$

To make (9) and (10) the same, (10) needs to be written in terms of X(0) and X(t). One can see that setting t=0 in (10) does not make the matrix 0 so one needs to pull out a 1/m from the first element of the second row of (10). This gives:

$$X(t)-X(0) = (1/m)\begin{vmatrix} i\cos(mt)\sin(mt) & \sin^2(mt) \\ -\sin^2(mt) & -i\cos(mt)\sin(mt) \end{vmatrix} \quad (11)$$

Expressions (9) and (11) are the same. Thus, for a particle at rest, there is one zitterbewegung. When the particle starts to move, then the Schrodinger zitterbewegung describes the center of mass X coordinate. This explains the linear $pH^{-1}t$ term in the Schrodinger X(t). We, however, seem to have another zitterbewegung in the cm of mass frame with its own X matrix which has nothing to do with linear motion. It should be noted that in the Schrodinger approach, there is no information about X(0). Setting t=0, gives X(0)=X(0). The matrix approach, however, gives an explicit form for X and shows how it operates on vectors.One might argue that the two zitterbewegung scenarios are the same, but this leads to the following problem. The X matrix can be given time dependence through exp(iHt)Xexp(-iHt). This matrix, however, will not give [H, X(t)] = V(t) like it should. In the case of p=0, there is still a momentum operator P(matrix in our one dimensional model) such that [M,P]=0 and one does not have [P,X]=1, only [p,x]=1.

In conclusion, we see that a one dimensional model of Einstein´s energy momentum equation (1) with 2x2 matrices (for energy eigenvalues of +E and -E) leads to two kinds of zitterbewegung formalisms. The first is related to the motion of the particle through space and forces one to impose [p,x]=1 (or x=d/dp if p is the momentum parameter) as there is no matrix X such that [H,X]=V where H is given by (2) and V by (3) for our one dimensional model. In the Dirac equation, one obtains V as a matrix by treating x as a variable and p=d/dx when performing [H,x]=V. In the center of mass frame, where p=0, there is, however, such an X matrix, and one can obtain a different zitterbewegung equation as given by (8). In the limit of p=0, the two become the same. It is possible that the center of mass frame contains more physical information about X than the full solution which involves a moving particle. Some effects of the internal motion on x, even if it is moving, appear in the Schrodinger zitterbewegung. We do not know if this issue has been dealt with in other models or if there is an explanation for V appear as a matrix in the Dirac equation while p and x do not, but rather appear as (d/dx and x).

References

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