Matrix Representations of the Relativistic Energy-Momentum Equation: A 2x2 Approach

Shihab Ahmed Al-Mutairi[1] & Dr. Hassan Ali Al-Qarni[2]

[1]Research Assistant, Petroleum Engineering Department, King Fahd University of Petroleum & Minerals, Dhahran, Saudi Arabia

[2]Professor, Petroleum Engineering Department, King Fahd University of Petroleum & Minerals, Dhahran, Saudi Arabia

In [2], it was argued that one could establish a 2x2 matrix eigenvalue equation from Einstein´s 1905 energy momentum equation $p^2 + m^2 = E^2$ (1) by assuming +E and -E and making a 2x2 matrix. The purpose of this note is to show that no such assumption is necessary. The 2x2 matrix approach can be shown to follow naturally from (1). (Note: In the Dirac derivation, an argument of linearization was made in order to obtain the Dirac equation from the Klein Gordon equation.) We also propose a different argument for obtaining a V (velocity) and P (momentum) matrix and examine properties of both. We also consider the time dependence of the V and P matrices and show that it is consistent with either helical motion or oscillatory motion in a dimension perpendicular to that of the motion of the particle. This oscillatory motion disappears if v=0 or p=0, but there seems to be a different oscillatory motion for p=0.

Matrix Formalism

To establish a matrix eigenvalue equation, one can begin by dividing (1) by E^2

$\cos^2(a) + \sin^2(a) = 1$ (2) where a is the angle appearing in the right angle triangle with E as the hypotenuse, p as the adjacent and m as the opposite, where m is the rest mass and c has been set to 1.

Now: $\cos(a) = \cos^2(a/2) - \sin^2(a/2)$ and $\sin(a) = 2\sin(a/2)\cos(a/2)$ so (2) can be rewritten as:

$$\cos(a)\,[\cos^2(a/2) - \sin^2(a/2)] + \sin(a)\, 2\sin(a/2)\cos(a/2) = 1 \quad (3)$$

This, in turn, can be written as the expectation value of an eigenvalue equation.

$$(\cos(a/2)\,,\ \sin(a/2))\left\{\cos(a)\begin{vmatrix} 1 & 0 \\ 0 & -1 \end{vmatrix}\begin{vmatrix}\cos(a/2) \\ \sin(a/2)\end{vmatrix} + \sin(a)\begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix}\begin{vmatrix}\sin(a/2) \\ \cos(a/2)\end{vmatrix}\right\} = 1 \quad (4)$$

This is equivalent to the energy eigenvalue equation listed in [2]:

$$\begin{vmatrix} p & m \\ m & -p \end{vmatrix} \text{ vector } = E \text{ vector} \quad (5)$$

where the positive energy eigenvector is (m, E-p) with the normalization factor: sqrt(2E(E-p)). This eigenvector can be rewritten as:

(1/sqrt(2)) (sqrt(1+v), sqrt(1-v)) (6) and v=cos(a) (for c=1)

Then, the eigenvector is also equivalent to (cos(a/2), sin(a/2)) which is the same eigenvector which appears in (4). Thus, no assumptions are made in developing a 2x2 matrix formalism of the energy momentum relation (1). The matrix in (5) has positive and negative energy eigenvalues, but no assumptions have been made forcing these, neither have any assumptions regarding ¨linearization¨ been made. What has been done specifically has been to rewrite cos(a) (which is proportional to v) and sin(a) in terms of the angles (a/2). Thus, one can see that v has been directly written as:

(cos(a/2), sin(a/2)) |1 0| |cos(a/2)| i.e. the expectation value of a matrix.
|0 -1| |sin(a/2) |

In Dirac formalism [1], this velocity matrix V=|1 0| was justified by arguing that [H,x]=V
|0 -1|

and using [p,x]=1. In [2], we argued that pv + m sqrt(1+v^2) = E and then identified a V matrix by comparison. Here, we see that the V matrix appears naturally in rewriting cos(a) (proportional to v) in terms of the angles (a/2). The important question is whether the matrix V has any physical significance. Certainly its expectation value has meaning (namely v), but the matrix V has +1 and -1 eigenvalues and may describe both positive and negative velocities. This is in keeping with the positive energy eigenvector which is (1/sqrt(2)) (sqrt(1+v), sqrt(1-v)). As noted in [2], this looks like a photon bouncing back and forth in an object moving with velocity v. We also saw in [2], that setting p=0 yielded a positive energy eigenvector of (1/sqrt(2)) (1,), which again contains positive and negative velocity eigenvectors. Thus, we start with a classical picture of a particle moving to the right with velocity v, and mathematically obtain an equivalent picture of forward and backward ¨internal¨ motion¨. As noted in [2], this may be describing a kind of particle wave duality.

Analysis of Momentum Operator

As noted in [2], it is possible to create a momentum operator by multiplying a velocity matrix by an E/m matrix. One can do this in either order, but the two are the same up to a sign. Another way to obtain the momentum operator directly is to consider:

p= E cos(a) = E ($\cos^2$(a/2) - $\sin^2$(a/2))

p = (cos(a/2), sin(a/2)) E | 1 0 | |cos(a/2)|
| 0 -1| | sin(a/2)|

At this point, one may note that E, which is a number could be placed either to the left or right of the V matrix which appears above. E can then be replaced with the H matrix to create a P operator. Thus, one could have HV or VH. The two do not commute, but the expectation value of [H,V] is 0, so it is as if they commute in terms of expectation values.

Thus, the momentum operator is written as:

P=|p m| (7) which has an expectation value of p with the +E eigenvector. For p=0, it has an
|m p| expectation value of 0 with the eigenvector (1/sqrt(2))(1,1).

For p>0 it has eigenvalues p+im and p-im and an expectation value of p with the +E eigenvector. The p+im eigenvector is (1/sqrt(2)) ((m-ip)/(p+im) , 1) and the p-im one is 1/sqrt(2)((-m-ip)/(p-im) , 1). These can be rewritten as: (-i, 1) and (i,1) or (1,i) and (1, -i) which are the momentum eigenvectors for the p=0 case as well with a normalization of 1/sqrt(2). Thus, going from p=0 to p>0, the eigenvalues change, but the eigenvectors remain unchanged. In addition, the eigenvectors are independent of E, p, v and m. (For the V matrix, neither the eigenvectors nor the eigenvalues change moving from p=0 to p>0. In addition, the eigenvectors are independent of E, p, v and m.)

(For the H matrix, the eigenvectors change from 1/sqrt(2) (1,1) to (cos(a/2), sin(a/2)) going from p=0 to p>0. It is interesting to note that 1/sqrt(2) is the root mean square value of cos(a/2) and sin(a/2). It seems that adding momentum p rotates the (1,1) vector.)

Given these eigenvalues, it is interesting to note that (1) is equivalent to E= |p+im|. (It is like a photon with complex momentum.)
It should be noted that the expectation value of the matrix P*P, with the +E eigenvector, does not yield p*p, but rather p*p - m*m. Thus, this is a somewhat strange matrix as it has been derived only in its linear form from an expectation value.
For p=0, the eigenvalues are +im and -im, which have the same magnitude as the mass (energy) itself. The corresponding p=0 eigenvectors are (1/sqrt(2))(1,i) and (1/sqrt(2))(1-i).

To get a sense of why the P operator has complex eigenvalues and entries, one can consider a geometrical interpretation. P is the product of two matrices, V and H. H is an orthogonal matrix which can at most rotate a unit vector. V on the other hand creates reflections about the ¨x¨ axis. Here, we are using the notation of ¨x¨ and ¨y¨ axes because we have a two dimensional problem. Consider a vector sitting at an angle of -a/2. Then, rotate it by a, so it is sitting at an angle of a/2. Then, reflect it about the x axis, so that it is back to an angle of -a/2.

For the p=0 case, one can consider the eigenvector (1,i) as having a value of 1 along the x axis and a value of 1 along the z axis (with this z axis attached to the x axis) (where i moves one into another dimension). The H matrix |0 m| scales by m and flips the coordinates.
|m 0|

So one has a value of 1 in the y direction followed by 1 in the z direction (now attached to the y axis). The matrix V then gives (i,-1) i.e. a reflection about the x axis. Finally, one multiplies by -i (or pulls out an i belonging to the eigenvalue.) Multiplying by -i reflects about the origin together with making each coordinate move in 90 degrees. One is back at (1,i).

Thus, a complex two vector appears to be like a four vector with real entries.
(2+i, 3+i) +> (2, 1, 3, 1). Multiplying the two vector by i appears to be like multiplying the 4-vector by a matrix with two diagonal blocks of $\begin{vmatrix}0 & -1\\ 1 & 0\end{vmatrix}$.

Momentum and Velocity Eigenvector Expressions

It is interesting to calculate some expectation values of momentum and velocity.

p=0 Case

In this case, the energy eigenvector is |m>= (1/sqrt(2)) (1,1) which can be written easily in terms of the V and P matrix eigenvectors.

|+m>= (1/sqrt(2)) { |v=1> + |v=-1>}

|+m>= (1/sqrt(2)) { (-i+1) |p=im> + (1+i) |p=-im> }

For p>0:

|+E> = cos(a/2) |v=1> + sin(a/2)|v=-1>

|+E>= 1/sqrt(2) { (-isin(a/2) + cos(a/2) |p+im> + (cos(a/2) + isin(a/2)) |p-im> }

Here |p=im>= |p+im> and |p=-im>=|p-im> because the momentum eigenvectors do not change as one moves from p=0 to p>0.

One may examine the momentum p+im case (weight and eigenvector as one moves from p=0 to p>0). In four vector notation, one goes from:
(1, -1, 1, 1) to (cos(a/2), -sin(a/2), sin(a/2), cos(a/2)).

This is equivalent to operating on the +E energy 2 vector by V to obtain the first two entries of the 4 vector and operating on the +E energy 2 vector by $\begin{vmatrix}0 & 1\\ 1 & 0\end{vmatrix}$ (the sqrt(1-v^2) matrix)

to obtain the last two entries of the 4 vector.

Time Dependence of Momentum and Velocity

In [2], it was shown that the time dependence of an operator could be written as exp(iHt) operator exp(-iHt). This relationship was then used to calculate zitterbewegung, yielding the same results as quantum mechanical calculations. Let us calculate the time dependence of P and V.

V(t) = V matrix - (i/E) sin(2Et) P matrix

Evaluating the expectation with the +E eigenvector yields: <V(t)>=v -(i/E) sin(2Et) p
or <V(t)> = v(1 - i sin(2Et))
This is interesting because one originally started with a v as an expectation value with no time dependence implied. Now, one is stating that the expectation over time (with evolution given by exp(-iHt)) oscillates in value. The time average of <V(t)> is still v and the real part has no time dependence, it is v as expected.The oscillation term, however, involves a complex number. It may be possible to interpret this as there existing an oscillating v in another dimension. An oscillation in an extra dimension is also equivalent to circular motion in two extra dimensions. (The shadow on the x axis of an object moving in a circle with origin (0,0) in the xy plane follows oscillatory motion.) Thus, one could try to interpret the above scenario as either helical motion or as a particle moving in one direction and oscillating in another. Kerson Huang has already suggested this in another model. Many authors have also already suggested helical motion,e.g. Barut and Zanghi, Hestenes etc.

P(t) = P matrix + (i/2E) sin(2Et) B

where $B = \begin{vmatrix} -2m^2 & 2mp \\ 2mp & 2m^2 \end{vmatrix} = 2mE \begin{vmatrix} -\sin(a) & \cos(a) \\ \cos(a) & \sin(a) \end{vmatrix}$

Evaluating the expectation with the +E eigenvector yields:
$\langle P(t)\rangle = p - (i/E^2) \sin(2Et)\, 2pm^2$

Again, the expectation of p is oscillating in time. This is also consistent with helical motion or an oscillation in another dimension. It should be noted that both <V(t)> and <P(t)> disappear for v=0 and p=0. Thus, this oscillatory behaviour seems to only be associated with motion. Perhaps it is related to the deBroglie wavelength and wave motion.

Let us now consider p=0 and find <V(t)> and <P(t)> for time evolution given by exp(-iMt) where M is H with p set to 0. We find:

$V(t) = \begin{vmatrix} -1 & 0 \\ 0 & 1 \end{vmatrix} \cos(2mt) + i \sin(2mt) \begin{vmatrix} -1 & -1 \\ 1 & 1 \end{vmatrix}$ Taking the expectation with (1/sqrt(2))(1,1) we find:

<V(t)> = 0 + 0 There is no expectation for this motion, but it may exist nevertheless as a kind of internal motion, perhaps related to spin. For example, taking expectation values with the +1 velocity eigenvector gives -1(cos(2mt) + i sin(2mt)) In fact:

V(t) = -V { cos(2mt) + i sin(2mt) } + i sin(2mt) P (where P is the momentum operator with p=0)

One interesting point is that we have simply used (1) Einstein´s energy momentum equation. We have made no assumptions about quantum mechanics or about bosons versus fermions. (The Dirac equation is assumed to apply to fermions.)

P(t) = cos(2mt) P - i sin(2mt) mV Again <P(t)> =0 but it seems that there may be some internal motion. (P is the momentum operator with p=0 namely $m\left|\begin{matrix}0 & 1\\ 1 & 0\end{matrix}\right|$)

Commutation Relations

It is interesting to consider some of the commutation values of H, P and V. Like in quantum mechanics p, E and v (scalars) are expectation values of operators. (In [2], we argued that quantum mechanics is directly contained in (1).) In the case of the expectation values, which are scalars, everything commutes. The operators H, P and V, we have found, however, do not commute. It is interesting to note, however, that the expectation values of the operators commute, so in terms of expectations it is as if the operators commute. Here the expectation value is taken with the +E eigenvector, namely (cos(a/2), sin(a/s)) where cos(a) = p/E. Thus,

<[H,P]>=0 <[H,V]>=0 and <[P,V]>=0 Let us consider explicitly one evaluation:

$$<[H, P]> = 2m < \left|\begin{matrix} m & -p \\ -p & -m \end{matrix}\right| > = (\cos(a/2), \sin(a/2)) \left|\begin{matrix} m\cos(a/2) - p\sin(a/2) \\ -p\cos(a/2) - m\sin(a/2) \end{matrix}\right| = m\cos(a) - p\sin(a)$$

=Esin(a)cos(a)-Ecos(a)sin(a) = 0

It is also interesting to note that [H,V]+=0 i.e. the two anticommute as does [H.P]+=0.

Conclusion

It has been shown that the 1905 Einstein energy momentum equation is equivalent to a formalism based on expectation values 2x2 matrices with (cos(a/2), sin(a/2)) being the positive energy eigenvector with cos(a)=p/E. This formalism in turn gives rise to matrix eigenvalue equations and matrix operators for H, V and P, i.e. energy, velocity and momentum, although these are somewhat strange operators. In [2] we argued that this matrix formalism is of the form of quantum mechanics with time evolution given by exp(-iHt). Here we show that the expectation value of the time dependent V and P matrices is consistent with either helical motion or motion in one direction with an oscillation in another dimension. We also examined some details of the momentum operator and examined commutation relations.

References

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