Challenges in Modeling Zitterbewegung Using the Schrödinger Equation

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In 1930, Schrodinger proposed the idea of Zitterbewegung. In [1], calculations of Zitterbewegung depend on [H,x]=V, where V is a velocity matrix obtained from the Dirac quantum mechanical equation, which assumes [x,p]=1. A problem arises,however, because the expectation value of [H,x] using energy eigenvectors is 0, while the expectation value of V, also using energy eigenvectors, should be v. In [2], we discussed a one dimensional model based on Einstein´s 1905 equation: $E^2 = p^2 + m^2$ (c=1) (1) using 2x2 matrices. We noted that there was no matrix X such that [H,X]=V, but that V nevertheless existed. We calculated zitterbewegung and obtained the same result as in [1]. Thus, there seems to be a problem. In this note, we attempt to provide a solution to this problem, re-examine what appears to be a linear matrix theory describing (1) but leads to a probabilistic model, and then calculate dispersions of P and X, to see if there is a relationship which could describe standing electron waves.

Problem with Zitterbewegung

As noted, in [1] [H,x]=V, but the expectation of the LHS with an energy eigenvector is 0 because H is self-adjoint, while the expectation of V, the velocity matrix, should be v. Nevertheless, in [1], a calculation of X(t)-X(0) is performed using: [H,x]=V, V(t)=exp(iHt) V exp(-iHt) and dV/dt= [H, V].

In [2], we used a one dimensional 2x2 matrix approach to solving (1). We used the Hamiltonian:

$$\begin{vmatrix} p & m \\ m & -p \end{vmatrix}$$ which has +E and -E eigenvalues with (m, E-p) being a +E eigenvector with 1/sqrt(2E(E-p)) being its normalization.

We were able to identify $V= \begin{vmatrix} 1 & 0 \\ 0 & -1 \end{vmatrix}$ (2) as the velocity operator without any quantum

mechanical relation [x,p]=1, by comparing an energy eigenvalue equation with pv + m sqrt(1+ v^2) =E. The expectation value of V with the +E energy eigenvector is v. We were, however, unable to find any 2x2 matrix X such that [H,X] = V, as shown in [3]. Nevertheless, we proceeded as in [1] and were able to find the same Zitterbewegung result as [1] (Schrodinger´s zitterbewegung) namely:

$$X(t)=X(0) + tp H^{-1} + .5 H^{-1} (V-pH^{-1}) (\exp(-2iHt) -1)$$

Calculating the dispersion of the zitterbewegung of [1] (and [3]), however, does not lead to interesting results because the expectation of H^{-1} $(V-H^{-1})$ with a +E eigenvector leads to 0, so $[X(t)-X(0)]^2 = (tpH^{-1})^2$

Zitterbewegung with p=0

In [3], it was shown that with p=0, H(called M) = $\begin{vmatrix} 0 & m \\ m & 0 \end{vmatrix}$, then there is a 2x2 matrix X

such that [H,X]=V, with V still being given by (2). In this case, the +E eigenvector is (1/sqrt(2))(1,1) and the expectation of V is 0. A general matrix for X was found in [3] to be:

$$X = aI + dM + \begin{vmatrix} 0 & 0 \\ 1/m & 0 \end{vmatrix} \quad (3)$$

where a and d are arbitrary constants and I is the identity matrix.

The expectation value of the last matrix of X is 1/m. In [3], it was shown that zitterbewegung in this case is:

$$X(t)= X(0) + \begin{vmatrix} i\cos(mt)\sin(mt) & \sin^2(mt) \\ -\sin^2(mt) & -i\cos(mt)\sin(mt) \end{vmatrix} \quad (3a)$$

It was also shown that the Schrodinger zitterbewegung [1], reduces to this same value for p=0. Thus, in the p=0 case, there is no zitterbewegung problem. It appears that there may be some kind of circular motion with velocity projections along the x axis and an axis perpendicular to x. Thus, in the center of mass frame, there may be some rotational motion which may account for spin as already pointed out by Kerson Huang in 1952, Barut and Zanghi (1984) and others.

Proposed Solution

The problem with the Schrodinger zitterbewegung is that the expectation of V is not 0. A way to remedy this is to consider an electron moving back and forth between two ¨walls¨. Then, there would be both -v and v. In the one dimensional model discussed in [1], we noted that there are two Hamiltonians:

$$\begin{vmatrix} p & m \\ m & -p \end{vmatrix} \quad (4) \qquad \begin{vmatrix} -p & -m \\ m & p \end{vmatrix} \quad (5)$$

These have the same +E, -E eigenvalues, but the velocity for each is the negative of the other, namely V1=-V2

For the first case, the expectation of V with the +E eigenvector is v. For the second case, the +E eigenvector is (m, E+p) with normalization 1/ sqrt[2 E (E+p)]. Now, one can choose V1 as the velocity operator and take the expectation value with 1/sqrt(2) [| +E p> + |+E -p>], i.e. the two eigenvectors. This will lead to a vanishing expectation value. To consider creating a X that will describe the situation, consider: [H of (4), X] where X is taken from the p=0 case, because this is known to give V1,

i.e. X= |0 0| Then, [H of (4), X]= |m 0| (1/m) and with p changed to -p
|1/m 0| |-2p -m|

I.e. [H of (5), X] = |m 0 | (1/m) . Averaging the two gives V1, the velocity matrix.
|2p m|

This, then suggests a way to use the X of p=0 in the p>0 situation (as long as the electron or nucleon is bouncing back and forth between two walls. As a result, dispersions can be calculated.

Note: One can perform similar calculations for an energy matrix of:
|m p | The X matrix again only exists in the m=0 case. It is a little different, but similar overall
|p -m| results seem to emerge.

Re-examination of the Matrix Approach To $p^2 + m^2 = E^2$

At this point, we wish to re-examine the matrix approach to the momentum energy quadratic equation (1). (The ideas here also seem to apply to the interpretation of the Dirac equation for the electron as very similar linear algebra is performed.) It can be noted, that the matrix eigenvalue equation:

|p m | |a| = E |a| (5a)
|m -p | |b| |b|

consists of two quadratic equations equivalent to (1). In other words, it appears that ¨linearization¨ is being performed, but quadratic equations are actually being employed. To see this, consider: a=m and b=E-p, then:

pa + mb = Ea and ma - pb =Eb. These are two scalar equations which are equivalent to the quadratic energy momentum equation (1). These can be rewritten in the matrix format of (5a) This looks linear and one can multiply the matrix on the left of (5a) and obtain a diagonal matrix with E^2 on the diagonal, but this matrix equation was obtained from quadratic equations to begin with. It should not really be looked on as being linear. This is why one can compare the above equation to pv + m sqrt($1+v^2$) = E and obtain a velocity matrix V. The matrix, with eigenvalues +1 and -1, indicates that one can have velocity in the positive or negative directions. In the original quadratic equation (1), one would consider two exclusive scenarios, a particle moving to the right or one moving to the left. Here, one is incorporating the two possible pictures into one, and so a probabilistic model is emerging. Thus, the energy eigenvector (a,b) can be written as a linear combination of a positive velocity eigenvector and a negative one. One does not know whether the particle is moving to the right or left. If one calculates the expectation value of V (the matrix) with a normalized (a,b), one obtains p/E. If p is positive, this yields a positive number, otherwise a negative number. A priori, one does not know which it is, so the model needs to allow for both. This is the only reason that the energy eigenvector is a combination of the two opposite direction velocity eigenvectors. One is ¨pretending¨ to work with a linear model, when one is actually using a quadratic equation, and so forward and backward values have to be considered for various operators of linear observables, such as velocity. These are built in through linear combinations of eigenvectors. This seems to explain why a probabilistic quantum model is appearing out of a classical momentum energy equation (1). It is next important to calculate dispersions, to see if this model yields predictions which differ from the scenario of two classical ¨balls¨ moving back and forth, trapped between two walls. (Experimentally, electrons form standing waves even when reflected from mirrors which are separated by distances much greater than atomic lengths.[8])

It is interesting to note that in the case of p=0, the energy matrix leads to a positive energy eigenstate of (1/sqrt(2)) (1,1), which is a sum of a positive velocity eigenvector plus a negative one. The expectation of the V matrix is 0. In a purely, classical picture, one would say that there is no motion. The parameter v would be 0 and there would be no V matrix. In this picture of positive and negative velocities, there is an alternative solution, namely a combination of positive and negative velocities which still average to 0. The question is whether this corresponds to physical reality. In quantum mechanics, taking this idea seriously lead to trapped objects always having positive and negative motion, i.e. no zero velocity.

Dispersion of p and x and Standing Waves

In quantum mechanics, dispersion values, i.e. the square root of the expectation of $| X\text{-}X(ave)|^2$ are calculated and used in uncertainty principles, such as the well-known dispersion(p) dispersion(x) <=1 (here hbar is set to 1). For p=0, we can calculate the dispersion for the momentum matrix and the X matrix with the energy eigenvalue 1/sqrt(2) (1,1).

For X we use (3). The last matrix has an expectation value of 1/m and a absolute dispersion of 1/2m. The first two matrices do not contribute.

Here we have used $|X\text{-}Xave|^2 = (1/m^2) \begin{vmatrix} -.5 & 0 \\ 1 & .5 \end{vmatrix}^2$

Out of interest, consider (3a):
X(t)= X(0) + matrix in (3a). Here, X(0) is the last matrix in (3).

Then: X(t) averaged over time yields $(1/m) \begin{vmatrix} 0 & .5 \\ .5 & .0 \end{vmatrix}$ which gives the same expectation value for X,

namely 1/2m.

For P= |p -m| one has, with p=0 |0 -m| (6) and an expectation of 0.

|m   p| |m   0|

We find for the dispersion: m.   Thus:

(dispersion x) (dispersion p) = m/2m =   ½ in the center of mass or for p=0.

It is interesting that we have found that product of the dispersions is a constant.
One can calculate the expectation of [P, X] to compare. It is ½ as well.

Case of p>0

Let us now consider the p>0 case.

For P, the expectation is p, not 0 in the p>0 case. The matrix P-Pave is, however, still (6)
and the dispersion is m again because (P-Pave)*(P-Pave)=m*m I, where I is the identity matrix.

For X, things are a little more complicated. We use the X matrix (the last matrix in (3)), but must consider a dispersion with both eigenvectors   (m, E-p)/ sqrt(2E(E-p)) and (m, E+p)/sqrt(2E(E+p))

We find that the expectation value of X in both cases is 1/2E.

Next, we calculate |-½ E   0| * itself   = |1/(4E^2)   0|
|1/m   -½ E| |-1/(mE)   1/(4E^2)|

Taking the expectation value with (m, E-p)/sqrt(2E(E-p)) one obtains   1/(2E^2).
The square root is the dispersion of X.

Thus, one sees that (dispersion x) (dispersion p) proportional to m/E. In the p=0 case, it is proportional to a number.

Given that P has a dispersion as well as X, it seems that an electron trapped between two walls, separated by a very small distance will exhibit a quantum mechanical type of distribution, in other words a standing wave. In quantum mechanics, the fact that the dispersion of p multiplied by the dispersion of x is a constant is sometimes given as evidence that p is proportional 1/L where L is the deBroglie wavelength. Here, we see that this kind of relationship already exists in the center of mass frame, something not normally considered in quantum mechanics, and then carries over to the p>0.

Conclusion

In conclusion, there appears to be a problem with Schrodinger zitterbewegung calculations because the expectation of [H,X]=0 in an energy eigenstate, while the expectation of V should be v. In [3], we also saw that it was not possible to find an matrix X that would satisfy [H,X]=V, although matrices for V and dV/dt could be found. In the scenario of p=0, there is no problem with the calculations, and a matrix X was found in [3]. We use this same X for the p>0 case here, but assume that one requires both forward and backward motion in order to bypass the zitterbewegung problem. We calculate dispersion values of P and X in both the p=0 (center of mass) and p>0 cases and show that the product is a number in both cases. This seems to imply a type of standing wave pattern for an electron or nucleon trapped in a small space between two walls and points to a relationship of p proportional to 1/L, where L is the wavelength. In such a case , there seems to be a physical reason as there appears to be forward and backward motion in the center of mass frame (i.e. the energy eigenstate is a combination of positive and negative V velocity eigenstates.) This carries over to the p>0 case. Many authors such as Huang, Barut and Zhanghi and Hestenes have described a model in which the electron moves in a helix which in turn gives rise to spin and a wavelength. We also see that the so-called linear matrix equation in p and m is really a quadratic equation which takes into account the possibility of positive and negative eigenvalues of operators such as V (velocity in the positive or negative direction), thus a probabilistic model emerges. In a classical picture, one would consider two completely separate scenarios, one for each eigenvalue direction. Here, a linear combination of both eigenvectors is ¨built in¨.

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