Bridging the Gap: Quantum Mechanical Insights into Classical Physics

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In a series of notes [2] to [4], we examined a 2x2 matrix model which is naturally incorporated into Einstein´s 1905 energy momentum equation $p^2 + m^2 = E^2$ (1). We showed that it was possible to obtain energy, velocity and momentum matrices, calculate expectation, dispersion values and anticommutation relations and establish exp(-iHt) as the time evolution operator. From this, we calculated zitterbewegung and the time dependence of other matrices. The purpose of this note is to see if this formalism, established for a relativistic model can also be applied to the nonrelativistic case. Several cases are considered. Given that special relativity holds in the ¨nonrelativistic¨ case, the later being an approximation, the same ideas should apply. Next, we try to apply the model to the case of a classical spring, which is related to the production of phonons.

In this case, we apply the mathematics of the model to the classical spring, not the model itself. It is found that one can obtain a velocity matrix and find an energy matrix with time evolution of exp(-iHt). Furthermore, it appears that time dependent operator calculations predict classical periodic motion with a period of sqrt(k/m), where k is the spring constant and m the mass.

Brief Review of A 2x2 Matrix Model Of the Momentum Energy Equation (1) [2], [3], [4], [5]

In a relativistic model, p/E= cos(a) and m/E= sin(a). Here we set c=1. This allows one to write: cos(a)=cos2(a/2) - sin2(a/2) and sin(a) = 2sin(a/2) cos(a/2). These can be rewritten as a matrix expectation equation using the 2-vector (cos(a/2), sin(a/2)). Thus (1) becomes

$$\begin{vmatrix} p & m \\ m & -p \end{vmatrix} \begin{vmatrix} \cos(a/2) \\ \sin(a/2) \end{vmatrix} = E \begin{vmatrix} \cos(a/2) \\ \sin(a/2) \end{vmatrix} \quad \text{for the positive E eigenvalue} \quad (2)$$

This model is similar to that of Dirac´s, but no assumption of quantum mechanics or linearization is made. The model follows directly from (1).

A velocity V matrix can be obtained as $\begin{vmatrix} 1 & 0 \\ 0 & -1 \end{vmatrix}$ (3) and a $1-v^2$ matrix as $\begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix}$. (4)

Geometrically, the velocity V matrix represents a reflection about the x axis and the $1-v^2$ matrix, a rotation by 90 degrees.

Eigenvectors and eigenvalues of these matrices can be found and expectation values calculated as well. Furthermore, one can show that this model has a time evolution given by exp(-iHt) where H is the energy matrix $\begin{vmatrix} p & m \\ m & -p \end{vmatrix}$.

Then, the time dependence of operators can be found using:
$O(t)=\exp(iHt)\, O\, \exp(-iHt)$, where O is an operator or matrix. Expectation values can be taken of these time dependent operators and periodic motion such as zitterbewegung can be calculated.

For equation (1), it is possible to show that there is periodic motion for <V(t)> perhaps related to the deBroglie wavelength. One can then examine the p=0 case, i.e. $H=\begin{vmatrix} 0 & m \\ m & 0 \end{vmatrix}$
and see that there is still periodic motion occuring, which may be related to spin. Kerson Huang, Barut and Zhanghi and Hestens among others consider models of the electron, such as a helical model, in which ¨internal¨ motion (i.e. non translational motion) accounts for the deBroglie wavelength and spin.

Time Evolution by exp(-iHt)

Let us consider the time evolution in more detail by considering cos(Et) - i sin(Et)/E H. Let us use

$H= \cos(a) \begin{vmatrix} 1 & 0 \\ 0 & -1 \end{vmatrix} + \sin(a) \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix}$ (Here we have divided through by E.)

Now, any unit vector can be written as (cos(n), sin(n)). Operating on such a vector yields:

$\cos(t) \begin{vmatrix} \cos(n) \\ \sin(n) \end{vmatrix} - i \sin(t) \begin{vmatrix} \cos(a-n) \\ \sin(a-n) \end{vmatrix}$ From the second term it can be seen that n=a/2 is an eigenvector as expected.

Consider the root mean square average i.e. $\begin{vmatrix} \cos(n) + \cos(a-n) \\ \sin(n) + \sin(a-n) \end{vmatrix}$

Writing cos(a-n) = cos(a)cos(n) + sin(a)sin(n) and sin(a-n)=sin(a)cos(n) - cos(a)sin(n)
and then rewriting cos(a)=cos(a/2)cos(a/2) - sin(a/2)sin(a/2) and sin(a)=2sin(a/2)cos(a/2) yields: (cos(a/2), sin(a/2) or the energy eigenvector.
The energy matrix is a rotation and one is taking a unit vector plus its rotation through an angle of a. Thus, the energy matrix H will convert any unit vector into the energy eigenvector.

Note on the Velocity Matrix

The expectation of the velocity matrix with the energy eigenvector, namely (cos(a/2), sin(a/2)) yields cos(a) which is the velocity i.e. p/E for the relativistic case. The velocity matrix was obtained in this manner in the first place. (In the Dirac equation it is obtained from [H,x], but no such relation is used here.) Now, the velocity matrix has a geometric interpretation, it involves a reflection about the x axis, where x is being used here to describe the first component of the two vectors used. It is interesting to note that the energy eigenvector contains cos(a/2) which in turn can be written as a matrix expectation of the V matrix with the vector (cos(a/4), sin(a/4). If the expectation with the energy eigenvector (cos(a/2), sin(a/2)) has physical meaning and the V matrix itself has physical meaning, one might assume that the expectation of the V matrix with other vectors could have meaning. Calculating V(t) one sees that there is oscillation. For the energy eigenvector one obtains the result $\langle V(t)\rangle = v(1 - i\sin(2Et))$

This same result, however, is obtained using other vectors such as (cos(a/4), sin(a/4)) etc. (One could repeat the process and describe cos(a/4) in terms of the V matrix and (cos(a/8), sin(a/8))).

Does this mean that there are oscillations building open oscillations to produce the final oscillation of the expectation value of V(t) with the energy eigenvector?

Interchange of sin(a) and cos(a)

In the 2x2 relativistic matrix approach, p/E has been identified with cos(a) and m/E, with sin(a) and an energy matrix or H of $\begin{vmatrix} p & m \\ m & -p \end{vmatrix}$ has been used. One may argue that cos(a) and sin(a) may be interchanged. This is true and leads to the energy matrix or H $\begin{vmatrix} m & p \\ p & -m \end{vmatrix}$ which has +E and -E as eigenvectors. Under this interchange, eigenvectors change, but it seems that the essence of the physics remains the same. For example, one may calculate zitterbewegung and obtain the same terms as before, but in different locations in a matrix which is associated with different basis vectors. Now, the +E energy eigenvector is 1/sqrt(2E(E-m) (p, E-m).

2x2 Matrix Model in a Nonrelativistic Case

We consider the case with no potential to see if there is any connection between the relativistic potential free model and a nonrelativistic one. In the nonrelativistic approximation:

E (energy) = m + .5 p^2/m where m is the rest mass and p=mv.

or K(kinetic energy) = .5 p^2/m

Method 1

If one takes the kinetic energy equation and sets p/m to cos(a), one can write an expectation value equation, but it will not yield an eigenvalue equation, so this approach is not taken.

$$K=.5/p \quad (\cos(a/2), \sin(a/2)) \begin{vmatrix} 1 & 0 \\ 0 & -1 \end{vmatrix} \begin{vmatrix} \cos(a/2) \\ \sin(a/2) \end{vmatrix}$$

Method 2

The relativistic relations p=Ecos(a) and m=Esin(a) should still hold, but cos(a) is understood to be very small i.e. a is near pi/2 and sin(a) is very close to 1. Alternatively, one can note that: $p^2 + m^2 = E^2$, even in the nonrelativistic case, if one drops high a p^4 term, so the cos(a), sin(a) relations apply.

Thus, the kinetic energy is a small correction to the mass energy. In such a case, one can write:

$E = p^2/2m + m$ or 1= .5/m p cos(a) + sin(a) which again leads to a matrix formulation by replacing cos(a) with $\cos^2(a/2)-\sin^2(a/2)$ and sin(a) with 2sin(a/2)cos(a/2). Thus:

$$1 = (\cos(a/2), \sin(a/2)) \begin{vmatrix} p/2m & 1 \\ 1 & -p/2m \end{vmatrix} \begin{vmatrix} \cos(a/2) \\ \sin(a/2) \end{vmatrix} \quad (3)$$ where p is small compared to 1.

One can now make an eigenvalue equation, but the eigenvalues are strange namely +/- (1/m) $\mathrm{sqrt}(p^2/4 + m^2)$ or +/- [1+ K/4m)] where K is the kinetic energy. . For p=0, one obtains +/- 1. The problem here is that eigenvalues are not transparently given in terms of the kinetic energy. In method 3, we will use an eigenvalue equation which explicitly has the positive energy kinetic energy as an eigenvalue. Given that ¨a¨ is near pi/2, so a is near pi/4 which means that cos(a/2) and sin(a/2) are very close to 1/sqrt(2). In other words, the energy eigenvector is close to 1/sqrt(2) (1,1) which is the p=0 solution. Since p/2m is considered to be very small, one would expect an eigenvector close to this value.

Now the parameter p/E can be associated with the matrix $\begin{vmatrix} 1 & 0 \\ 0 & -1 \end{vmatrix}$ because

$$p/E = (\cos(a/2), \sin(a/2) \begin{vmatrix} 1 & 0 \\ 0 & -1 \end{vmatrix} \begin{vmatrix} \cos(a/2) \\ \sin(a/2) \end{vmatrix}$$ m/E can similarly be associated with $\begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix}$

The eigenvalues of p/E (which is v, velocity, to a first approximation) are +1 and -1 with eigenvectors (1,0) and (0,1). The m/E operator and eigenvalues +1 and -1 and eigenvectors (1/sqrt(2)) (1,1) and (1/sqrt(2)(1,-1).

As in the relativistic case [2], the time evolution operator should be exp(-iHt), with H given by:

$$\begin{vmatrix} p/2m & 1 \\ 1 & -p/2m \end{vmatrix}.$$

This should evolve the velocity eigenvector into the energy eigenvector:
exp(-iHt) = cos(t) - i sin(t) H if one considers $1+ .25(p/m)^2 = 1$ so operating on (1,0) yields:

cost(t) (1,0) - i sin(t) (p/2m, 1) where p is very small compared to 1. Taking the root mean square averages leads to: (1+p/2m, 1) which is very close to (1, 1) because p is very small compared to 1. In fact, (cos(a/2) , sin(a/2) is proportional to (1+p/m, 1). In the relativistic case, one has exact time evolution, but in the nonrelativistic one of method 2.

Method 3

Perhaps the most transparent method is to consider the original relativistic energy eigenvalue equation and then replace E with m+.5/m p^2. The first term can then be combined with the energy matrix to give an equation for K (kinetic energy) as the eigenvalue. Thus:

|p-m m| |m | = .5/m p^2 |m | where E= m +.5/m p^2
|m -p-m| |E-p| |E-p | (4)

This equation has eigenvalues -m +/- E or K and -2m-K

(Note: In nonrelativistic calculations, one can still use $E^2=p^2+m^2$.)
The form of the +E energy eigenvalue does not change moving from the relativistic to nonrelativistic case. The normalization is given by 1/sqrt(2E(E-p)) so the eigenvector for positive energy is (cos(a/2), sin(a/2)) where a is very close to pi/2 or the eigenvector is close to 1/sqrt(2) (1,1).

For the evolution operator, H now has an extra exp(-imt)= exp(-imIt) (where I is the identity matrix) which is a phase. For the time evolution of operators and zitterbewegung, however, exp(iHt) V exp(-iHt) yields the exact results as the relativistic case (with p here now being the nonrelativistic momentum) because one can drop the -mI from the H of method 3 because the -mI piece drops out. Here we have assumed: exp[i (H-mI) t] = exp(iHt) exp(-mIt) where I is the identity matrix.

Again in the nonrelativistic case, there is no X matrix such that [H,X] = V, where the velocity V matrix is still given by:

|1 0|
|0 -1|

For the case of p=0, the energy eigenvalues are 0 and-2m. In the relativistic case they were m and -m and so one can see a shift by m. The eigenvectors remain the same. One can calculate zitterbewegung as in the relativistic case and see that there is periodic motion in the cm of mass case, i.e., there is still spin-like motion.

The energy matrix is this approach has an extra term of -mI (in comparison with relativistic case). This term commutes with the velocity matrix, so V should have the same time dependence.
Time evolution should still be given by exp(-iHt) because the extra term in the energy matrix is -mI, where I is the identity matrix which commutes with any matrix.

In (4) it is possible to set p=0. In such a case, H the energy matrix is m|-1 1|
|1-1 |.

The p/E (almost V) operator does not vanish in this case, and may indicate that there is motion in the center of mass, which might be spin motion. There is no reason that spin should only appear in the relativistic case, and not in the nonrelativistic one.

Analysis of a Classical Spring

The classical equation of a spring is Energy = .5 m v^2 + k x^2
A solution, as is well known, is: x=A cos(wt) where w=sqrt(k/m). Here A is the amplitude.

The solution for x and v depends on time, so let us consider the root mean square average of this equation so it contains only time independent parameters. The classical spring equation is of the same mathematical form as Einsein´s 1905 energy momentum equation (1) if one takes rms values of p and x .This raises a question. Can one apply that same mathematical arguments to this equation and obtain physical results? It seems that the same math can be applied, because the equation is of the same form as (1). Physically, however, one is not directly applying the same theory as in the relativistic quantum mechanical case because here p is proportional to cos(a) and x to sin(a). In the relativistic scenario, p/E was equal to cos(a) and m/E was equal to sin(a). Thus, these are physically different case. The idea of matrices, commutation, expectation values and time evolution under exp(-iHt), however, should all still hold. This is important, because these are quantum mechanical like calculations that normally would never be applied to a classical problem. In addition, one can calculate zitterbewegung and see if it has any possible physical meaning.

As noted, we will write p and x in terms of rms values for v and x. The amplitude cancels from both sides of an energy equation, but k, the spring constant, and m remain.

E= $(vrms)^2$ + $(xrms)^2$

with E= k/m.

This yields and energy equation H= | (vrms) (xrms) |
| (xrms) -(vrms) |

Which has eigenvalues of +/- sqrt[$(vrms)^2$ + $(xrms)^2$]. The positive energy unnormalized eigenvector is ((xrms), sqrt(E) - (vrms)). These again yield (cos(a/2), sin(a/2) as the normalized eigenvector. Time evolution should still be governed by exp(-iHt) as one has the same mathematical values as in the relativistic quantum mechanical model.

Thus, V(t), the time dependent velocity = exp(iHt) V exp(-iHt).

or V(t) = V + i sin(2sqrt(E) t)/sqrt(E) | 0 -(xrms) | - $\sin^2$ [sqrt(E) t)] 2sin(a) |-sin(a) cos(a)|
|(xrms) 0 | |cos(a) sin(a) |

The imaginary term disappears if one takes expectation values with either the energy eigenvector or the velocity eigenvector. The term $\sin^2[\text{sqrt}(E)\ t]$ can be rewritten in terms of cos[2sqrt(E) t] as is usually done in zitterbewegung calculations [1]. Here E is sqrt(k/m) and the 2 will disappear if one considers that there is a positive velocity eigenvector and a negative one. One switches from one to the other and the time for each is 1/2sqrt(E). This leads to an overall period of 1/sqrt(E).

Thus, it seems that quantum mechanical like calculations, including time evolution and zitterbewegung can yield classically relevant results. Barut and Zhanghi develop a model in which they also calculate a classical zitterbewegung [8]. This is important as one needs to see physical evidence for zitterbewegung. Recently, there have been several experiments which have detected zitterbewegung motion [13],.

From [1] and [4], we see that X(t) zitterbewegung is given by:

$$X(t)=X(0) + tp\ H^{-1} + .5\ H^{-1}\ (V-pH^{-1})\ (\exp(-2iHt) -1)$$

This leads also to periodic motion with a frequency of sqrt(k/m), which is expected if the velocity operator is also behaving in a similar manner. Thus, it seems that there may some physical relevance to zitterbewegung calculations if they are predict physical motion in the classical regime. Alternatively, one can calculate X(t) by using $\begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix}$ as the matrix associated with position.

This yields:

$$X(t)= X(0) - i\cos(a)\sin[2\text{sqrt}(E)\ t]\ X(0) + \sin^2[\text{sqrt}(E)\ t]\ 2\cos(a) \begin{vmatrix} \sin(a) & -\cos(a) \\ -\cos(a) & -\sin(a) \end{vmatrix}$$

The last term disappears if one takes the expectation value with (cos(a/2), sin(a/2)), the energy eigenvector yielding a sin[2sqrt(E) t] time dependence for X with sqrt(E)= sqrt(k/m). This periodic function is out of phase with that for velocity as expected.

Quantum Mechanical Like Behaviour in the Classical World

It has long been known that sound waves and light waves form standing waves in the classical world (i.e. on classical length scales), and also exhibit diffraction and interference. In 1995, it was suggested [12] that the density of a rotating classical gas in a cylinder, governed by statistical mechanics, could also be modeled using a wavefunction, which would also yield the density. In such a case, there would be physical oscillations (phonons) which could maintain the density and the forces due to rotation in the gas. Recently, French researchers [16] were able to create interference patterns for a classical particle riding on a physical wave (e.g. consider a tiny bead on a vibrating stereo speaker).J WM Bush [15] has carried out numerous experiments and analyzed much quantum like behaviour of these particles riding on waves. In addition, Prof. G.

Fuchs has performed experiments using sound to flip the spin of electrons. ¨ Fuchs' group demonstrates control of a quantum phenomenon using classical vibrations. ¨[13]

Conclusion

In conclusion, we see that we can apply the relativistic 2x2 matrix approach in the nonrelativistic regime, but a number of issues arise.We see that although it is straightforward to convert cos(a) into a 2x2 matrix expectation value equation using cos2(a)-sin2(a), it may not always be possible to convert it into an eigenvalue equation (method 1). Even if it can be made into an eigenvalue equation, the eigenvalues may be strange (method 2). There does, however, appear to be a way to obtain an eigenvalue equation which uses the nonrelativistic kinetic energy as an eigenvalue and seems to preserve the ¨quantum mechanical¨ features of the relativistic model. In addition, it appears that one can mathematically apply these quantum mechanical techniques (e.g. evolution operator, matrices, expectation values etc) to a classical spring equation and seemingly obtain physical results for time dependent operator equations. This is interesting because it implies that quantum mechanical formalism may hold in classical physics.
Some recent experiments, especially those of JWM Bush [15] have shown ¨quantum mechanical like behaviour¨ in classical situations.

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