

## Operator-Differential Equations of Second Order: Solvability and Multiple Characteristic Analysis

Ali M. Al-Ansari\*

\*Associate Professor, Department of Mechanical Engineering, University of Bahrain, Isa Town, Bahrain

### ABSTRACT

The paper studied the matters of solvability of the second order operator differential equations with multiple characteristics on the semi-axis. In the space of Sobolev type  $W_2^2(R_+; H)$  identified conditions that ensured the unique and well-posed solvability are of the considered equation which is cleared by its operator coefficients

**KEYWORDS:** solvability condition, boundary-value problem, Hilbert space, self-adjoint operator, Sobolev type  $W_2^2(R_+; H)$ .

### INTRODUCTION

In a separable Hilbert Space  $H$ , we consider a polynomial operator pencil of the second order

$$P(\lambda) = (\lambda E + A)^2 + \lambda A_1, \quad (1)$$

Where  $E$  is the identity operator,  $A$  is a self-adjoint positively defined operator, and  $A_1, A_2$  are linear operator unbounded. This equation can associate a polynomial pencil

$$Q\left(\frac{d}{dt}\right)u(t) = f(t) \quad (2)$$

In space  $L_2(R; H)$ , we denote by  $L_2(R; H)$  (see. [1]) Hilbert space of all vector valued functions defined in  $R_+ = [0, +\infty)$  with values in  $H$ , which have the norm

$$\|f\|_{L_2(R; H)} = \left( \int_0^{+\infty} \|f(t)\|_H^2 dt \right)^{1/2} < \infty.$$

Now we introduce the following set:

$$W_2^2(R; H) = \left\{ u(t) : \frac{d^2 u(t)}{dt^2} \in L_2(R; H), A^2 u(t) \in L_2(R; H) \right\}.$$

With the norm

$$\|u\|_{W_2^2([a, b]; H)} = \left( \left\| \frac{d^2 u}{dt^2} \right\|_{L_2([a, b]; H)}^2 + \|A^2 u\|_{L_2([a, b]; H)}^2 \right)^{1/2},$$

### MAIN RESULTS

**Definition 1.** If  $f(t) \in L_2(R; H)$  there exists vector valued function  $u(t) \in W_2^2(R; H)$ , satisfying equation (2) almost everywhere, then it is called a regular solution of equation (2).

**Definition 2.** If for any  $f(t) \in L_2(R; H)$  there exists a regular solution  $u(t)$  of equation (2), and holds the inequality.

$$\|u\|_{W_2^2(R; H)} \leq \text{const} \|f\|_{L_2(R; H)}$$

Then we will call the equation (2) regularly solvable.

In the paper we will get the conditions, expressed in the operator coefficients of equation (1), we proved the regular solvability of equation (2).

Let's introduce the following notations. Calling  $Q_0$  and  $Q_1$  the operators acting from the space  $W_2^2(R; H)$  as follows:

$$Q_0 u(t) \equiv \left( \frac{d}{dt} + A \right)^2 u(t), u(t) \in W_2^2(R_+; H)$$

$$Q_1 u(t) \equiv B \frac{du(t)}{dt}, u(t) \in W_2^2(R_+; H)$$

**Theorem 1.** The operator  $Q_0$  is an isomorphism from the space  $W_2^2(R; H)$  to the space  $L_2(R; H)$

**Proof.** Taking into account the theorem of intermediate derivatives [2], it is easy to prove that the operator  $Q_0$  acts from  $w_2^2(R; H)$  to  $L_2(R; H)$  be bounded. Using Fourier transforms for the equation  $Q_0 u(t) = f(t)$ ,  $f(t) \in L_2(R; H)$ ,  $u(t) \in w_2^2(R_+; H)$ , we obtain

$$(i\zeta E + A)^2 \tilde{u}(\zeta) = \tilde{f}(\zeta),$$

where  $\tilde{u}(\zeta)$ ,  $\tilde{f}(\zeta)$  Fourier transform for the functions  $u(t)$ ,  $f(t)$ , respectively the operator pencil  $(i\zeta E + A)^2$  is invertible and moreover

$$\tilde{u}(\zeta) = (i\zeta E + A)^{-2} \tilde{f}(\zeta). \quad (3)$$

Hence,

$$u(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} (i\zeta E + A)^{-2} \left( \int_0^{+\infty} f(s) e^{-i\zeta t} ds \right) e^{i\zeta t} d\zeta, \quad t \in \mathbb{R}.$$

we show that  $u(t) \in w_2^2(R_+; H)$ . by using the parseval equality and (3), we obtain

$$\begin{aligned} \|u\|_{w_2^2(R_+; H)}^2 &= \left\| \frac{d^2 u}{dt^2} \right\|_{L_2(R_+; H)}^2 + \|A^2 u\|_{L_2(R_+; H)}^2 = \\ &= \left\| -\zeta^2 u(\xi) \right\|_{L_2(R_+; H)}^2 + \left\| A^2 u(\xi) \right\|_{L_2(R_+; H)}^2 = \\ &= \left\| -\zeta^2 (i\zeta E + A)^{-2} f(\zeta) \right\|_{L_2(R_+; H)}^2 + \left\| A^2 (i\zeta E + A)^{-2} f(\zeta) \right\|_{L_2(R_+; H)}^2 \leq \\ &\leq \sup_{\zeta \in \mathbb{R}} \left\| -\zeta^2 (i\zeta E + A)^{-2} \right\|_{H \rightarrow H}^2 \|f(\zeta)\|_{L_2(R_+; H)}^2 + \\ &+ \sup_{\zeta \in \mathbb{R}} \left\| A^2 (i\zeta E + A)^{-2} \right\|_{H \rightarrow H}^2 \|f(\zeta)\|_{L_2(R_+; H)}^2 \end{aligned} \quad (4)$$

From the spectral decomposition of the operator  $A$  (denoting by  $\sigma(A)$  the spectrum of operator  $A$ ) for  $\zeta \in \mathbb{R}$  we have

$$\begin{aligned} \left\| -\zeta^2 (i\zeta E + A)^{-2} \right\| &= \sup_{\mu \in \sigma(A)} \left| -\zeta^2 (i\zeta + \mu)^{-2} \right| \leq \\ &\leq \sup_{\mu \in \sigma(A)} \frac{\zeta^2}{\zeta^2 + \mu^2} \leq 1, \end{aligned} \quad (5)$$

$$\begin{aligned} \left\| A^2 (i\zeta E + A)^{-2} \right\| &= \sup_{\mu \in \sigma(A)} \left| \mu^2 (i\zeta + \mu)^{-2} \right| \leq \\ &\leq \sup_{\mu \in \sigma(A)} \frac{\mu^2}{\zeta^2 + \mu^2} \leq 1. \end{aligned} \quad (6)$$

Taking into account (5) and (6) into (4) we obtain :

$$\|u\|_{w_2^2}^2 \leq \|f(\zeta)\|_{L_2(R; H)}^2 = 2 \|f(t)\|_{L_2(R_+; H)}^2$$

. Consequently,  $u(t) \in W_2^2(R_+; H)$ . We finish the proof of the theorem by taking into account the Banach theorem of the inverse operator.

**Theorem 2.** Let the operators  $A_j A^{-j}$ ,  $j = 1, 2$  are bounded on  $H$ . Then The operator  $Q_1$  acting from the space  $W_2^2(R_+; H)$  to  $L_2(R_+; H)$  is bounded.

**Proof.** Since  $u(t) \in W_2^2(R_+; H)$  then by applying the theorem on intermediate derivatives [2], we have

$$\begin{aligned} \|Q_1 u\|_{L_2(R_+; H)} &\leq \|A_1 A^{-1}\|_{H \rightarrow H} \left\| A \frac{du}{dt} \right\|_{L_2(R_+; H)} \leq \\ &\leq \text{const} \|u\|_{W_2^2(R_+; H)} \end{aligned}$$

The theorem is proved

Theorem (1) show that the norm  $\|Q_1 u\|_{L_2(R_+; H)}$  equivalent in  $W_2^2(R_+; H)$  to the norm  $\|u\|_{W_2^2(R_+; H)}$  therefore, by theorem of intermediate derivatives [2] the number  $N_1 = \sup_{0 \neq u \in W_2^2(R_+; H)} \left\| A^j \frac{d^{3-j} u}{dt^{3-j}} \right\|_{L_2(R_+; H)} \|Q_0 u\|_{L_2(R_+; H)}^{-1}$  is finite.

Let us estimate these values

**Theorem 3.**

$$N_1 \leq c_1 \text{ where } c_1 = \frac{1}{2}$$

**Proof**

Let  $Q_0 u(t) = f(t)$  and using the Fourier transform we obtain

$$\|A(i\xi)(i\xi E + A)^{-2} \tilde{f}(\xi)\|_{L_2(R_+; H)} \leq \sup_{\xi \in R} \|A(i\xi)(i\xi E + A)^{-2}\|_{H \rightarrow H} \|\tilde{f}(\xi)\|_{L_2(R_+; H)} \quad (7)$$

Taking into account the spectral decomposition of operator  $A$ , we estimate for  $\xi \in R$  the following norms:

$$\begin{aligned} \|A(i\xi)(i\xi E + A)^{-2}\| &= \sup_{\mu \in \sigma(A)} |\mu(i\xi)(i\xi + \mu)^{-2}| \leq \\ &\leq \sup_{\substack{t = \frac{\xi^2}{2} \geq 0 \\ \mu}} \frac{1}{t+1} = c_1 \end{aligned} \quad (8)$$

Substituting from (8) in (7), we have

$$\|A(i\xi)(i\xi E + A)^{-2} \tilde{f}(\xi)\|_{L_2(R_+; H)} \leq c_1 \|\hat{f}(\xi)\|_{L_2(R_+; H)},$$

Are equivalent in its turn to the inequalities

$$\left\| A \frac{du}{dt} \right\|_{L_2(R_+; H)} \leq c_1 \|Q_0 u\|_{L_2(R_+; H)},$$

The theorem is proved.

**Theorem 4.** Let the operators  $A_1 A^{-1}$  defined in Theorem 3. Then equation inequality  $c_1 \|A_1 A^{-1}\|_{H \rightarrow H} < 1$ , Where  $c_1$  is defined in Theorem 3. Then the equation (2) regularly solvable.

**Poof .** Clearly that equation (2) can be Written as

$$Q_0 u(t) + Q_1 u(t) = f(t), \quad (9)$$

Where  $f(t) \in L_2(R_+; H)$ ,  $u(t) \in W_2^3(R_+; H)$ . The operator  $Q_0$  by Theorem (1) there exists a bounded inverse, which acts from  $L_2(R_+; H)$  within  $W_2^2(R_+; H)$ . Then after replacing  $Q_0 u(t) = v(t)$  equation (9) can be written as

$$(E + Q_0 Q_0^{-1})v(t) = f(t).$$

Now we prove under the theorem conditions that the norm  $\|Q_1 Q_0^{-1}\|_{L_2(R_+; H) \rightarrow L_2(R_+; H)} < 1$ .

By theorem (3), we have

$$\begin{aligned} \|Q_1 Q_0^{-1} v\|_{L_2(R_+; H)} &= \|Q_1 u\|_{L_2(R_+; H)} \leq \|A_1 A^{-1}\|_{H \rightarrow H} \left\| A \frac{d u}{dt} \right\|_{L_2(R_+; H)} \leq \\ &\leq \sum_{j=1}^2 c_{1j} \|A_1 A^{-1}\|_{H \rightarrow H} \|Q_0 u\|_{L_2(R_+; H)} = c_{11} \|A_1 A^{-1}\|_{H \rightarrow H} \|v\|_{L_2(R_+; H)} \end{aligned}$$

Consequently,

$$\|Q_1 Q_0^{-1}\|_{L_2(R_+; H) \rightarrow L_2(R_+; H)} \leq c_{11} \|A_1 A^{-1}\|_{H \rightarrow H} < 1.$$

Thus, the operator  $E + Q_1 Q_0^{-1}$  is invertible in  $L_2(R_+; H)$  and hence  $u(t)$  can be determined by  $u(t) =$

$$Q_0^{-1} \left( E + Q_1 Q_0^{-1} \right)^{-1} f(t), \text{ moreover}$$

$$\|u\|_{W_2^2(R_+; H)} \leq \|P_0^{-1}\|_{L_2(R_+; H) \rightarrow W_2^2(R_+; H)} \left\| \left( E + Q_1 Q_0^{-1} \right) \right\|_{L_2(R_+; H) \rightarrow L_2(R_+; H)} \|f\|_{L_2(R_+; H)} \leq \text{const} \|f\|_{L_2(R_+; H)}.$$

The theorem is proved.

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